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Generalized Semi-Infinite Programming: on generic local minimizers

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Abstract In this paper a basic structural problem in Generalized Semi-Infinite Programming is solved. In fact, under natural and generic assumptions we show that at any (local) minimizer the "Symmetric Reduction Ansatz" holds.

Keywords Semi-Infinite Programming · Optimality condition · Reduction Ansatz

AMS Subject Classification 90C34 · 90C46 · 90C31 · 90C47

1 Introduction and the main result

Generalized semi-infinite optimization problems have the form

$$GSIP\colon \text{minimize}\, f(x) \quad \text{s.t.} \quad x \in M$$

with

$$M \;=\; \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y(x)\}$$

and

$$Y(x) \;:=\; \{y \in \mathbb{R}^m \mid g_k(x, y) \leq 0,\; k = 1, \ldots, s\}.$$

The defining real valued functions $f, g_k, k = 0, \ldots, s$, are assumed to be d times, $d \geq 2$, continuously differentiable.

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Among various other applications [4,10], semi-infinite problems frequently arise in approximation theory, particularly in Chebyshev approximation. In fact, the approximation of a continuous function F on a nonempty and compact set $Z \subset \mathbb{R}^M$ by a member of the family of continuous functions $\{a(p, \cdot)|\ p \in P\}$ with some parameter set $P \subset \mathbb{R}^N$, leads to the nondifferentiable problem

$$\min_{p \in P} ||F(\cdot) - a(p, \cdot)||_{\infty, Z} \ = \ \min_{p \in P} \max_{z \in Z} |F(z) - a(p, z)| \, .$$

It is not hard to see that the latter problem can be rewritten as the semi-infinite optimization problem

$$\min_{(p,q) \in P \times \mathbb{R}} q \quad \text{s.t.} \ \pm (F(z) - a(p, z)) \ \le \ q \ \text{ for all } \ z \in Z.$$

The main advantage of this reformulation is that the semi-infinite problem is a *smooth* optimization problem (if all defining functions are smooth), whereas the original problem is intrinsically nonsmooth. The price to pay for smoothness is, of course, the presence of infinitely many inequality constraints. Solution methods for this specially structured semi-infinite problem can be found, for example, in [5].

In engineering applications such as the approximation of a thermo-couple characteristic or the construction of low pass filters in digital filtering theory, a modification of the Chebyshev approximation problem, termed *reverse Chebyshev approximation*, is considered [6,9]. In this framework, let F be a continuous function on a nonempty and compact set $Z(q) \subset \mathbb{R}^m$ which depends on a parameter $q \in Q$. Given an approximating family of functions $a(p, \cdot)$ and a desired precision $e(p, q)$, the aim is to find parameter vectors p and q such that the domain $Z(q)$ is as large as possible without exceeding the approximation error $e(p, q)$. This yields the problem

$$\max_{(p,q) \in P \times Q} Vol(Z(q)) \ \text{ s.t. } \ ||F - a(p, \cdot)||_{\infty, Z(q)} \ \le \ e(p, q),$$

where $Vol(Z(q))$ denotes the M–dimensional volume of $Z(q)$. Again, this intrinsically nonsmooth optimization problem can be reformulated with semi-infinite constraints. However, as opposed to the above situation in Chebyshev approximation, now one obtains a *generalized* semi-infinite optimization problem:

$$\max_{(p,q) \in P \times Q} Vol(Z(q)) \ \text{ s.t. } \ \pm (F(z) - a(p, z)) \ \le \ e(p, q) \ \text{ for all } \ z \in Z(q),$$

since the index set of inequality constraints depends on the decision variable. Numerical approaches to this problem class for small dimensions are presented in [6] and [9]. For an introduction to generalized semi-infinite programming and a different numerical solution approach see [10].

For the subsequent analysis, let the space of d times continuously differentiable functions be endowed with the strong C^c-topologies, $c \in \{0, \ldots, d\}$. In the strong C^c-topology, a base neighborhood of a function $\bar{f}$ is given by $U_\varepsilon(\bar{f})$, where ε is a strictly positive continuous "distance function". The neighborhood $U_\varepsilon(\bar{f})$ consists of all functions f such that the modulus of the difference of the function values and all partial derivatives up to order c of the functions f and $\bar{f}$ at any point x are smaller than $\varepsilon(x)$. Note that the C^c-topologies get stronger with increasing c, i.e. C^c-open sets are also C^{c+1}-open and C^c-dense sets are also C^{c-1}-dense. Note that the space of d times continuously differentiable functions endowed with the strong C^d-topology constitutes a Baire space. We say that a set is *generic* if it contains a countable intersection of C^d-open and dense subsets. Generic sets in a Baire space are dense as well.

Recall that a set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called locally bounded if for each $\bar{x} \in \mathbb{R}^n$ there exists a neighborhood U of $\bar{x}$ such that $\bigcup_{x \in U} W(x)$ is a bounded subset of $\mathbb{R}^m$. The following Assumption A is a usual assumption in Generalized Semi-Infinite Programming.

Assumption A The mapping $Y : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally bounded.

Let $\mathcal{A}$ denote the set of problem data $(f, g_0, \ldots, g_s)$ such that Assumption A is satisfied. The set $\mathcal{A}$ is C^0-open [8].

It is well-known that the feasible set M need not be closed, therefore we consider the closure $\overline{M}$ of M instead. In [2] we have recently found a representation of $\overline{M}$ in explicit terms. In fact, for C^d-generic problem data from $\mathcal{A}$, the closure of the feasible set is given by

$$\overline{M} = \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y^<(x)\} \tag{1}$$

with

$$Y^<(x) = \{y \in \mathbb{R}^m \mid g_k(x, y) < 0,\ k = 1, \ldots, s\}. \tag{2}$$

The minimization problem of f on the closure of M will be referred to as $\overline{GSIP}$. Now we are ready to state our main result.

Theorem 1 **(Main result)** *There exists a C^d-generic set of problem data $\mathcal{Y}$ such that for all $(f, g_0, \ldots, g_m) \in \mathcal{A} \cap \mathcal{Y}$ the following assertions hold.*

(*i*) *For $\overline{GSIP}$, any local minimizer is a Karush–Kuhn–Tucker point.*
(*ii*) *At any (local) minimizer of $\overline{GSIP}$ the Symmetric Reduction Ansatz holds.*

Note that—by continuity—a local minimizer of $GSIP$ is automatically a local minimizer of the problem $\overline{GSIP}$, thus the assertion of Theorem 1 also holds for $GSIP$. The precise definitions of a Karush–Kuhn–Tucker point in the context of Generalized Semi-Infinite Programming and of the Symmetric Reduction Ansatz can be found in Sect. 2. Section 3 is devoted to the proof of Theorem 1.

2 Preliminaries

2.1 Karush–Kuhn–Tucker points

In order to describe $\overline{M}$ in a symmetric way, let us consider the following families of sets, where $K := \{0, \ldots, s\}$:

$$N(x) := \{y \in \mathbb{R}^m \mid g_k(x, y) \leq 0,\ k \in K\},$$
$$N^<(x) := \{y \in \mathbb{R}^m \mid g_k(x, y) < 0,\ k \in K\}.$$

Definition 2 We say that the Mangasarian Fromovitz Constraint Qualification (shortly MFCQ) holds at $y \in N(x)$ if there exists an open halfspace in $\mathbb{R}^m$ containing all the partial gradients $D_y g_k(x, y)$, $k \in K_0(x, y)$, where $K_0(x, y) := \{k \in K \mid g_k(x, y) = 0\}$ stands for index set of active constraints.

Remark 3 Note that MFCQ is violated at $y \in N(x)$ if and only if $0 \in \text{conv}\{D_y g_k(x, y) \mid k \in K_0(x, y)\}$, where the symbol conv represents the convex hull.

Lemma 4 *Let Assumption A be satisfied and let $\bar{x}$ be a boundary point of M. Then the following two assertions are true*

(*i*) *The set $N(\bar{x})$ is nonvoid.*
(*ii*) *MFCQ is violated for $N(\bar{x})$ at any of its points.*

Proof ad (*i*)*:* Since $\bar{x}$ lies in the boundary of M it must be an limit point of the complement $\complement M$ of M. By definition we have $x \in \complement M$ if and only if there exists some $y(x) \in Y(x)$ with $g_0(x, y(x)) < 0$. Letting $x \in \complement M$ tend to $\bar{x}$, the corresponding points $y(x)$ remain in a certain compact set by Assumption A. Thus, after choosing the points x tending to $\bar{x}$ accordingly, we may assume that the points $y(x)$ converge, say to $\bar{y}$. By continuity of the problem data it holds $\bar{y} \in Y(\bar{x})$ and $g_0(\bar{x}, \bar{y}) \leq 0$, i.e. $\bar{y} \in N(\bar{x})$.
ad (*ii*)*:* For an indirect proof assume that (ii) does not hold. By (i), the set $N(\bar{x})$ is nonvoid, i.e. there exists at least one $\bar{y} \in N(\bar{x})$, where MFCQ holds. Now MFCQ implies that there exists $y \in N^<(\bar{x})$. By continuity this implies $y \in N^<(x)$ for x from an entire neighborhood U of $\bar{x}$, but this, in view of the argumentation in (i), yields $U \subset \complement M$. Thus $\bar{x}$ does not lie in the boundary of M, a contradiction. □

From [7] we know that the following assumption holds for a C^1-open and C^d-dense set $\mathcal{B}$ of problem data.

Assumption B For any $x \in \mathbb{R}^n$ and $y \in N(x)$ the set of gradients $\{Dg_k(x, y) \mid k \in K_0(x, y)\}$ is linearly independent.

What makes Lemma 4 valuable for us, in order to state first order optimality conditions, is the following crucial observation. In fact, under Assumption B, the violation of MFCQ at $y \in N(x)$ is equivalent with $V(x, y) \neq \emptyset$, where $V(x, y)$ is the compact convex subset of $\mathbb{R}^n$ defined by the following equality in $\mathbb{R}^n \times \mathbb{R}^m$:

$$V(x, y) \times \{0\} \;=\; \left(\mathbb{R}^n \times \{0\}\right) \;\cap\; \operatorname{conv}\{Dg_k(x, y) \mid k \in K_0(x, y)\}. \tag{3}$$

Remark 5 By Assumption B, $0 \in \mathbb{R}^n \times \mathbb{R}^m$ does not belong to the affine hull of the vectors $\{Dg_k(x, y) \mid k \in K_0(x, y)\}$, thus $0 \in \mathbb{R}^n$ is not contained in the affine hull of $V(x, y)$.

For x from the boundary of $\overline{M}$ and y from $N(x)$, the elements of $V(x, y)$ serve as a substitute for the gradients of active inequality constraints used for stating the Karush–Kuhn–Tucker Condition for standard nonlinear optimization problems in $\mathbb{R}^n$. Keeping this in mind, the concept of a "Karush-Kuhn-Tucker Condition" for points satisfying the following (optimality) condition shall be motivated.

Definition 6 (KKT–SIP) We say that the Karush–Kuhn–Tucker (shortly KKT–SIP) Condition is satisfied for $\overline{GSIP}$ at $x \in \overline{M}$ and x is called a Karush–Kuhn–Tucker (shortly KKT) point for $\overline{GSIP}$ if the following holds. There exist a finite (possibly empty) set of parameters $\{y^1, \ldots, y^p\} \subset N(x)$, vectors $v^i \in V(x, y^i)$ and multipliers $\mu^i \geq 0, i = 1, \ldots, p$, such that

$$Df(x) \;=\; \sum_{i=1}^{p} \mu^i \cdot v^i.$$

(Note that the KKT condition reads as $Df(x) = 0$ if $N(x) = \emptyset$.)

In order to make sure that the KKT–SIP Condition is a necessary optimality condition, one needs constraint qualifications. In virtue of [2], there exists a C^d-generic set of problem data $\mathcal{C}$, such that for problem data from $\mathcal{A} \cap \mathcal{C}$ the following form of a linear independence constraint qualification is satisfied at any feasible point.

Definition 7 (LICQ–SIP) We say that the *Linear Independence Constraint* Qualification (shortly LICQ–SIP) holds at x from $\partial\overline{M}$ if for any finite subset $\{y^1, \ldots, y^p\} \subset N(x)$ and any choice of $v^i \in V(x, y^i), i = 1, \ldots, p$, the vectors $\{v^1, \ldots, v^p\}$ are linearly independent. If x belongs to the interior of M, then we also say that LICQ–SIP is satisfied.

For later use recall from [10] that a local minimizer x for $\overline{GSIP}$ is necessarily a KKT point provided that the Assumptions A and B are satisfied, LICQ holds at x and MFCQ holds on $Y(x)$.

2.2 The Epigraph reformulation

From now on we assume that the Assumptions A and B are satisfied and, moreover, LICQ–SIP holds on $\overline{M}$. In other words, we simply consider problem data from $\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}$. In virtue of [2], under these assumptions, $\overline{M}$ is described by (1), (2). As in [3], we may then consider the epigraph reformulation for describing the feasible set $\overline{M}$. The latter reformulation is based upon the fact that, according to (1), (2), $x \in \overline{M}$ if and only if for each $y \in \mathbb{R}^m$ it holds $g_0(x, y) \geq 0$ or $y \notin Y^<(x)$. Therefore, we have $x \in \overline{M}$ if and only if for any $y \in \mathbb{R}^m$ the maximal entry among the function values of $g_k, k \in K$, at (x, y) is non-negative. The latter is equivalent with $\psi(x) \geq 0$, where $\psi(x)$ denotes the optimal value of the problem

$$Q(x): \min_{(y,z)\in\mathbb{R}^m\times\mathbb{R}} z \quad \text{s.t.} \quad g_k(x, y) - z \leq 0,\ k \in K. \tag{4}$$

Assumption A ensures that for local considerations the set $\mathbb{R}^m$ in (4) can be replaced by an appropriate compact set. This implies the optimal value function ψ to be well-defined. Since ψ constitutes a continuous function, we necessarily have $\psi(x) = 0$ for points x from the boundary of $\overline{M}$. For $x \in \partial\overline{M}$, the points (y, z) with $y \in N(x)$ are exactly the global minimizers of $Q(x)$.

Note that the defining functions of $Q(x)$ are of class C^d. Thus we may define y to be a *nondegenerate* element of $N(x)$, if $(y, 0)$ is a nondegenerate minimizer of $Q(x)$ in the sense of Jongen/Jonker/Twilt [7]. The latter means, that the linear independence constraint qualification, strict complementary slackness and the second order sufficiency condition hold at the solution $(y, 0)$ of $Q(x)$ (for details see [7]).

Definition 8 (Symmetric Reduction Ansatz) At $\bar{x} \in \overline{M}$ the Symmetric Reduction Ansatz is said to hold if all elements of $N(\bar{x})$ are nondegenerate.

Note that, if $\psi(\bar{x}) > 0$, we have $N(\bar{x}) = \emptyset$, thus here the Symmetric Reduction Ansatz holds trivially. For $\psi(\bar{x}) = 0$, if the Symmetric Reduction Ansatz holds, for any $y^i \in N(\bar{x})$ there exist functions $z^i \in C^d(U)$, defined on an open neighborhood U of $\bar{x}$, such that

$$\overline{M} \cap U = \{x \in U \mid z^i \geq 0 \text{ for all } i \text{ such that } y^i \in N(\bar{x})\},$$

see [3] for details. The functions z^i are called the *reducing* functions.

Remark 9 Note that the Symmetric Reduction Ansatz at $x \in \overline{M}$ implies the following assertions.

(i) Each of the functions z^i is the optimal value function of the problem $Q(.)$, being localized to a small neighborhood of y^i.
(ii) $V(x, y^i)$ is a singleton, its point is a positive multiple of $D^\top z^i$. This motivates, why we consider the elements of $V(x, y^j)$ as substitutes for the gradients of the active inequality constraints.

3 Proof of the main result

3.1 Proof of assertion (i)

At a free local minimizer x of the function f (belonging to int M) the derivative Df vanishes, and therefore x is necessarily a KKT-point for $\overline{GSIP}$. We only need to prove that also local minimizers from $\partial\overline{M}$ are KKT-points, provided that the problem data are chosen accordingly.

In order to analyze local minimizers from the boundary of $\overline{M}$, we assume that the problem data are chosen from $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$, hence $\overline{M}$ can be written in a the form used in Sect. 2.2, based on the problems $Q(x)$. In fact, the feasible set of $\overline{GSIP}$ consists of all x such that the optimal value $\psi(x)$ of $Q(x)$ is non-negative. This is again the standard description of a feasible set in Generalized Semi-Infinite Programming, using one more variable (namely z) than the original description. However, essential for attaining a first order necessary optimality condition, MFCQ holds trivially at any point of the feasible set of the problem $Q(x)$.

Since MFCQ holds for $Q(x)$, a local minimizer x must be a KKT point, of course for the $Q(x)$-description of the set $\overline{M}$, see the citation of [10] at the end of Subsect. 2.1. For the $Q(x)$-description the set corresponding to $N(x)$ reads as

$$\{(y, z) \in \mathbb{R}^m \times \mathbb{R} \mid g_k(x, y) - z \le 0,\ k \in K,\ z \le 0\},$$

and for any pair (y, z) from this set we have $z = 0$, since x is a boundary point of $\overline{M}$ (implying $\psi(x) = 0$). The set corresponding to $V(x, y)$, now $V(x, y, z)$, consists of all vectors $v \in \mathbb{R}^n$ such that there exist nonnegative λ_k, $k = 0, \dots, s+1$, such that at (x, y, z) we have

$$\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} = \sum_{k \in K_0(x,y)} \lambda_k \begin{pmatrix} D_x g_k \\ D_y g_k \\ -1 \end{pmatrix} + \lambda_{s+1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This is exactly the set $V(x, y)$ known from the original formulation. Hence the KKT condition for the $Q(x)$-description implies the KKT condition for the original description. Since $\mathcal{B} \cap \mathcal{C}$ is C^d-generic, this proves Assertion (i) of the theorem. Note that the set $\mathcal{Y}$ in Theorem 1, defined in Subsect. 3.2, is a subset of $\mathcal{B} \cap \mathcal{C}$.

3.2 Proof of assertion (ii)

Our argumentation is based on jet representations of the Karush–Kuhn–Tucker Condition in Definition 6. Since the Symmetric Reduction Ansatz trivially holds for points from the interior of $\overline{M}$, we only have to check minimizers $x \in \partial M$. We suppose that the problem data are from $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$, which implies that x is a KKT of $\overline{GSIP}$. Writing $N(x) = \{y^1, \dots, y^p\}$ with some $p \in \{1, \dots, n\}$ and representing any of vectors $v^i \in V(x, y^i)$ by means of a linear combination with strictly positive multipliers of a minimal number of vectors $v^{i,j}$ forming vertices of the polytope $V(x, y^i)$, we see that there exist numbers $l_i, r^i \in \mathbb{N}$, and index sets $K^i, L^{i,j} \subset K$, $i \in \{1, \dots, p\}$, $j \in \{1, \dots, l_i\}$ such that:

(i) For any $i \in \{1, \dots, p\}$ we have $K_0(x, y^i) = K^i$.
(ii) For any $i \in \{1, \dots, p\}$ the cone generated by $\{D_y g_k(x, y^i) \mid k \in K^i\}$ has dimension equal to r^i.
(iii) For any $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, l_i\}$, we have $L^{i,j} \subset K^i$. Moreover, the set of partial gradients $\{D_y g_k(x, y^i) \mid k \in L^{i,j}\}$ generates a cone of dimension $|L^{i,j}| - 1$ and the convex hull of the set of full gradients $\{D g_k(x, y^i) \mid k \in L^{i,j}\}$ contains the vector $(v^{i,j}, 0)$.

(iv) The vector $Df(x)$ belongs to the cone generated by the vectors $\{v^{i,j} \mid i = 1, \ldots, p, j = 1, \ldots, l_i\}$.

Note that the vectors $v^{i,j}$ are uniquely determined. Letting $l := \sum_{i=1}^{p} l_i$ and

$$\Upsilon := \{(x, y^1, \ldots, y^p, v^{1,1}, \ldots, v^{p,l_p}) \in \mathbb{R}^n \times \mathbb{R}^{mp} \times \mathbb{R}^{nl} \mid (i) - (iv) \text{ hold}\},$$

we first show the existence of a C^d-generic set $\mathcal{D}$ of problem data, such that for problem data from $\mathcal{A} \cap \mathcal{Y}$ with $\mathcal{Y} := \mathcal{B} \cap \mathcal{C} \cap \mathcal{D}$, the set Υ consists of isolated points only.

Note that in the definition of Υ, the variables $v^{i,j}$ play a structurally different role than the variables x and y^i. In such situations the structured jet transversality theorem (see [1]) may be applied, providing the existence of a C^d-generic set $\mathcal{D}$ of problem data such for all problem data from $\mathcal{Y}$, defined above, the subset in the so-called jet-space, characteristic for Υ, is met transversally. This means in particular, that Υ constitutes a stratified set whose dimension coincides with the difference between the amount of available degrees of freedom and the number of independent equations representing the Conditions (i)–(iv) defining Υ.

The ambient space of Υ has dimension $n+mp+nl$. Now let us count the loss of freedom caused by the conditions defining Υ. For each $i \in \{1, \ldots, p\}$ the Condition (i) stands for $|K^i|$ equations. For nonvoid Υ we obviously have only $r^i \leq m$ and $r^i < |K^i|$. Hence, Condition (ii) reduces the freedom by $(m - r^i)(|K^i| - r^i)$ degrees. Condition (iii) causes for any (i, j) a loss of freedom of $r^i + 1 - |L^{i,j}|$ degrees just for demanding that the set of partial gradients $\{D_y g_k(x, y^i) \mid k \in L^{i,j}\}$ generates a cone of dimension $|L^{i,j}| - 1$ only. The condition that the latter partial gradients contain the origin in their convex hull does not reduce the degree of freedom. However the fact that $v^{i,j}$ is uniquely determined by the corresponding full gradients, reduces the freedom by as many as n degrees. For summing up, we distinguish two cases.

Case 1 ($r^i = m$) In this case the loss of freedom $Loss_i$ caused by (i–iii) at (x, y^i), sums up to

$$Loss_i = |K^i| + 0 + n \cdot l_i + \sum_{j=1}^{l_i} (r^i + 1 - |L^{i,j}|). \tag{5}$$

Setting $M_i := \max_j |L^{i,j}|$ on easily sees

$$|K^i| \geq M_i + l_i - 1. \tag{6}$$

Using (6) for estimating $|K^i|$ in (5) and M_i for estimating $|L^{i,j}|$, we have

$$\begin{aligned} Loss_i &\geq M_i + l_i - 1 + n \cdot l_i + l_i(m + 1 - M_i) \\ &= (l_i - 1)(m + 1 - M_i) + m + 1 + l_i - 1 + n \cdot l_i \\ &\geq m + l_i + n \cdot l_i. \end{aligned} \tag{7}$$

All inequalities in (7) turn to equalities if only if $|L^{i,j}| = m + 1$ for all j. For avoiding additional slack in (6), one must have $K^i = L^{i,1} \cup \cdots \cup L^{i,l_i}$.

Case 2 ($r^i < m$) Setting (for convenience) $\alpha_i := |K^i| - r^i - 1 \geq 0$, one has

$$Loss_i = |K^i| + (m - r^i)(|K^i| - r^i) + n \cdot l_i + \sum_{j=1}^{l_i} (r^i + 1 - |L^{i,j}|). \tag{8}$$

Using (6) for estimating $|K^i|$ in (8) and M_i for estimating $|L^{i,j}|$, we have

$$\begin{aligned} Loss_i &\geq M_i + l_i - 1 + (m - r^i)(1 + \alpha_i) + n \cdot l_i + l_i(r^i + 1 - M_i) \\ &\geq m + l_i + n \cdot l_i + (m - r_i)\alpha_i + (l_i - 1)(r^i + 1 - M_i) \\ &\geq m + l_i + n \cdot l_i. \end{aligned} \tag{9}$$

For having equalities at all places in (9), one necessarily has $\alpha_i = 0$ and $(l_i - 1)(r^i + 1 - M_i) = 0$, i.e. $|K^i| = r^i + 1$ and $l_i = 1$. The latter implies $L^{i,1} = K^i$, since otherwise there would be positive slack in (6).

Finally, we see that Condition (iv) reduces the freedom by $n - l$ degrees. If $Df(x)$ does not belong to the relative interior of the cone considered in (iv), this causes additional loss of freedom. Summarizing, Conditions (i)-(iv) cause a loss of at least

$$\sum_{i=1}^{p}(m + l_i + n \cdot l_i) + n - l = mp + nl + n$$

degrees of freedom, i.e. for problem data from $\mathcal{A} \cap \mathcal{Y}$ the set Υ has dimension at most 0. Since another loss of freedom would cause Υ to be empty, we are in the following situation:

In Case 2, the point y^i is a nondegenerate element of $N(x)$, i.e. it delivers the constraint $z^i(x) \geq 0$ for locally describing $\overline{M}$. In Case 1, however, the situation may become more involved. The point $(y^i, 0)$ still must be a strongly stable local minimizer of $Q(x)$. But now, the local optimal value function ψ takes the form

$$\psi = \max_J \psi^J, \tag{10}$$

where J runs through the set $\mathcal{J}$ of all minimal index sets $J \subset K_0(x, y^i)$ such that the convex hull of the partial gradients $\{D_y g_k(x, y^i) \mid k \in J\}$, contains the origin of $\mathbb{R}^m$, and where ψ^J is the optimal value function of the problem

$$Q^J(x): \qquad \min_{(y,z)\in\mathbb{R}^m\times\mathbb{R}} z \quad \text{s.t.} \quad g_k(x, y) - z \leq 0, \; k \in J.$$

Due to the choice of problem data from $\mathcal{A} \cap \mathcal{Y}$ and $\Upsilon \neq \emptyset$, the point $(y^i, 0)$ is a nondegenerate minimizer of $Q^J(x)$, implying that ψ^J is a C^d-function. Returning to the Karush-Kuhn-Tucker Condition, we see that the index sets $L^{i,j}$ belong to $\mathcal{J}$. For $J = L^{i,j}$, the gradient $D\psi^J$ is a positive multiple of $v^{i,j}$. By LICQ-SIP and Remark 5, all the vectors $v^{i,j}$, $i \in \{1, \ldots, p\}$, $i \in \{1, \ldots, l_i\}$, are linearly independent. This linear independence, together with the fact that $Df(x) = \sum_{i,j} \mu^{i,j} \cdot v^{i,j}$ with all multipliers $\mu^{i,j}$ being positive, and with the maximum type of ψ in (10), implies that $l_i = 1$ for all $i \in \{1, \ldots, p\}$. Otherwise there would exist a strictly feasible strict descent direction for $\overline{GSIP}$, contradicting the local minimality of x.

Remark 10 At this point we have proved that at a local minimizer the feasible set does not have a disjunctive structure, see [10].

Altogether, we have $\psi = \psi^J$ with $J = K^i$ in Case 1, hence $(y^i, 0)$ is a nondegenerate minimizer of $Q(x)$ in Case 1, too.

Remark 11 We conjecture that the C^d-generic set $\mathcal{Y}$ in Theorem 1 can even be chosen C^2-open.

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